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BRAUER–SEVERI SCHEMES OF FINITELY GENERATED ALGEBRAS

BY

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ABSTRACT

In this paper we characterize the Brauer–Severi scheme of a fixed degree (as defined by M. van den Bergh) of a finitely generated algebra over a commutative ring as the Proj of a graded commutative ring.

Introduction

The purpose of this paper is to give an alternate characterization of the Brauer–Severi scheme of a finitely generated algebra as defined by M. van den Bergh in [9]. We do this by relating the Brauer–Severi scheme to the variety of representations of the algebra as defined below. In particular, for any finitely generated algebra A over a commutative ring R we show that its Brauer–Severi scheme of degree n (n a positive integer) is isomorphic to $\mathrm{Proj}(Q_{A,n})$ for some graded commutative R-algebra $Q_{A,n}$. The graded R-algebra $Q_{A,n}$ is shown to be generated by a subset of semi-invariants of the diagonal GL_n action on the fibered product of the scheme of representations of A of rank n with $\mathbb{A}^n$. This generalizes Corollary 1.11 of [8] to finitely generated algebras over an arbitrary commutative ring.

Here we will assume that all rings will be associative rings with an identity element and all ring homomorphisms will preserve the identity elements. Let us choose a commutative base ring k. We will use the definition of a k-scheme

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found in [4, §I.1.9] so that a k-scheme will be a functor from commutative k-Algebras to Sets with certain additional properties. For any positive integer m let $F_m = k\{\mathcal{Y}_1, \dots, \mathcal{Y}_m\}$ be the free (noncommutative) k-algebra on m generators. If R is a commutative k-algebra, we will let $RF_m = R \otimes_k F_m$.

We will also need to make use of the functor M_n from commutative k-Algebras to k-Algebras defined by letting $M_n(S)$ be the ring of $n \times n$ matrices with entries in the commutative k-algebra S. If $f\colon S \to T$ is a homomorphism of commutative k-algebras, then the homomorphism $M_n(f)\colon M_n(S) \to M_n(T)$ will be defined by applying f to each entry of each matrix in $M_n(S)$.

If Q is a graded k-algebra and $h \in Q$ is a homogeneous element, we will adopt the following notations: Q_h will denote the localization of the ring Q at the multiplicatively closed subset $\{1, h, h^2, \dots\}$ and $Q_{((h))}$ will denote the sub-k-algebra of Q_h generated by the homogeneous elements of degree zero in Q_h.

Now fix a commutative k-algebra R and a positive integer n. For any finitely generated R-algebra we define the **Lattice Representation Scheme of degree** n to be $\mathrm{LRep}_n(A) = \mathrm{Spec}(S_{A,n})$ where we use $S_{A,n}$ to denote the universal commutative R-algebra (uniquely determined up to isomorphism) given in [1]. If $\rho_{A,n}\colon A \to M_n(S_{A,n})$ is the corresponding R-algebra homomorphism, we have the following **universal property**:

> Given any R-algebra homomorphism $\phi\colon A \to M_n(T)$ where T is a commutative R-algebra, there exists a **unique** R-algebra homomorphism $\eta\colon S_{A,n} \to T$ such that $\phi = M_n(\eta) \circ \rho_{A,n}$.

We will call a pair $(S_{A,n}, \rho_{A,n})$ a **universal pair for A of degree n.**

One consequence of this universal property is that for any commutative R-algebra T, there is a natural one-to-one correspondence between the set of R-algebra representations $\phi\colon A \to M_n(T)$ and the set of T-valued points $\mathrm{LRep}_n(A)(T)$. For any R-algebra representation $\phi\colon A \to M_n(T)$, let η_ϕ denote the T-valued point of $\mathrm{LRep}_n(A)$ corresponding to ϕ.

For any positive integer m, it follows from [1] that we can choose $S_{RF_m,n}$ to be $S_{m,n} = R[x_{i,j}^{(\ell)} | 1 \leq i,j \leq n, 1 \leq \ell \leq m]$, the polynomial ring in the mn^2 commuting indeterminants $x_{i,j}^{(\ell)}$ and we can choose $\rho_{RF_m,n}$ to be $\rho_{m,n}\colon RF_m \to M_n(S_{m,n})$ where $\rho_{m,n}$ is determined by $\rho_{m,n}(\mathcal{Y}_\ell) = [x_{i,j}^{(\ell)}]$ for all $1 \leq \ell \leq m$. Therefore, given a surjection $\tau\colon RF_m \to A$, the universal property of $(S_{m,n}, \rho_{m,n})$ determines an embedding of $\mathrm{LRep}_n(A)$ in $X_{m,n} = \mathrm{LRep}_n(RF_m)$ as a closed subscheme. (Note that, as a scheme, $X_{m,n}$ is just the affine scheme $\mathbb{A}^{mn^2}$.)

Finally, we define a GL_n-action on $\mathrm{LRep}_n(A)$ as follows. For any commutative

R-algebra T and for any $\eta \in \mathrm{LRep}_n(A)(T)$, $\gamma \in \mathrm{GL}_n(T)$ let $\eta^\gamma\colon S_{A,n} \to T$ be the R-algebra homomorphism corresponding to the representation given by

$$\begin{aligned} \phi\colon A &\longrightarrow M_n(T) \\ a &\longmapsto \gamma(M_n(\eta) \circ \rho_{A,n}(a))\gamma^{-1}. \end{aligned}$$

Note that if η, η' are T-points of $\mathrm{LRep}_n(A)$ then η and η' induce isomorphic T-lattice representations (i.e., the A-module structures induced on T^n by η and η' are isomorphic) if and only if there exists an $\gamma \in \mathrm{GL}_n(T)$ such that $\eta' = \eta^\gamma$.

1. The Brauer-Severi scheme

Let R be a commutative k-algebra and let A be an R-algebra that is not necessarily commutative. Let $B_n(A,R)$ denote the set of all pairs (φ, P) such that P is a left A-module that is a finitely generated projective R-module of constant rank n and $\varphi\colon A \to P$ is a surjective A-module homomorphism. We will call two pairs (φ, P) and (ψ, Q) **equivalent** if there exists an A-module isomorphism $u\colon P \to Q$ such that $u \circ \varphi = \psi$. In this case, we will write $(\varphi, P) \sim (\psi, Q)$ to indicate the pairs are equivalent.

Let $Bsev_n(A,R)$ denote the set of equivalence classes of $\sim$ in $B_n(A,R)$. Then we let $\mathrm{Bsev}_n(A,R)$ denote the functor from commutative R-Algebras to Sets that takes the commutative R-algebra S to the set $Bsev_n(A \otimes_R S, S)$. The functor $\mathrm{Bsev}_n(A,R)$ naturally extends to a functor on R-schemes and is a closed subfunctor of the Grassmannian functor, hence is an R-scheme (see [9, Prop. 2]). So we define the **Brauer-Severi scheme of A over R of degree n** to be the R-scheme $\mathrm{Bsev}_n(A,R)$. Now we can use the following lemma of M. Van den Bergh's to relate the Brauer-Severi scheme of degree n to the R-lattice representation scheme of degree n.

LEMMA 1.1 ([9, Lemma 3]): *Let R be a commutative ring, T be a commutative k-algebra, and A an arbitrary R-algebra. Then the T-points of $\mathrm{Bsev}_n(A,R)$ are in one-to-one correspondence with equivalence classes of triples (ϕ, x, P) where P is a finitely generated projective T-module of constant rank n, $\phi\colon A \to \mathrm{End}_T(P)$ is a k-algebra homomorphism such that $\phi(R) \subseteq T$, and $x \in P$ is such that $\phi(A)Tx = P$.*

In the above lemma, we say two triples (ϕ, x, P) and (ϕ', x', P') representing T-points of $\mathrm{Bsev}_n(A,R)$ are **equivalent** if there exists an $A \otimes_R T$-module isomorphism $u\colon P \to P'$ such that $u(x) = x'$.

Note: If T is a commutative k-algebra and the triple (ϕ, x, P) represents a T-point of $\mathrm{Bsev}_n(A,R)$, then by Lemma 1.1 the representation ϕ induces an R-algebra structure on T. Furthermore, if two triples represent the same T-point, then the two triples induce the same R-algebra structure on T. Therefore, once we specify a T-point of $\mathrm{Bsev}_n(A,R)$ we have specified an R-algebra structure on T.

For every commutative R-algebra T identify T^n with $\mathbb{A}^n_R(T)$. Then to every triple (ψ, z, T^n) that represents a T-point of $\mathrm{Bsev}_n(A,R)$ we can associate a T-point (η_ψ, z) of $\mathrm{LRep}_n(A) \times_R \mathbb{A}^n_R$, where η_ψ is the unique homomorphism such that $\psi = M_n(\eta_\psi) \circ \rho_{A,n}$.

More generally, assume T is a commutative R-algebra and the triple (ϕ, x, P) represents a T-point of $\mathrm{Bsev}_n(A,R)$. Then there exists a faithfully flat commutative T-algebra T' such that $P \otimes_T T'$ is a free T'-module of rank n (for example, let $T' = \prod_{\mathfrak{m}} T_{\mathfrak{m}}$ where the product ranges over the maximal ideals $\mathfrak{m}$ of T). Choose a T-module isomorphism β: $P \otimes_T T' \to (T')^n$ and let $\tilde{\beta}$: $\mathrm{End}_{T'}(P \otimes_T T') \to M_n(T')$ be the corresponding T-algebra isomorphism. Since T' is faithfully flat over T, we can identify P with a sub-T-module of $P \otimes_T T'$ and we can identify $\mathrm{End}_T(P)$ with its image in $\mathrm{End}_{T'}(P \otimes_T T')$ under the map $\phi \mapsto \phi \otimes \mathrm{id}_{T'}$. Then the triple $(\tilde{\beta} \circ \phi, \beta(x), (T')^n)$ represents a T'-point of $\mathrm{Bsev}_n(A,R)$. So, given this choice of T' and β, we can associate the T'-point $(\eta_{\beta,\phi}, \beta(x))$ of $\mathrm{LRep}_n(A) \times_R \mathbb{A}^n_R$ to the triple (ϕ, x, P).

This association of a T'-point of $\mathrm{LRep}_n(A) \times_R \mathbb{A}^n_R$ to (ϕ, x, P) depends of course on the choice of T' and on the isomorphism β. The question then arises, given a triple (ϕ, x, P) representing a T-point of $\mathrm{Bsev}_n(A,R)$ how are the various representatives of this triple in $\mathrm{LRep}_n(A) \times_R \mathbb{A}^n_R$ related? First, let us fix a faithfully flat commutative T-algebra T' such that $P \otimes_T T'$ is a free T'-module of rank n. Let β, β': $P \otimes_T T' \to (T')^n$ be T'-module isomorphisms. Then there exists a $\gamma \in \mathrm{GL}_n(T')$ such that $\beta' = \gamma \circ \beta$. Therefore, for any $f \in \mathrm{End}_{T'}(P \otimes_T T')$ we get $\tilde{\beta}'(f) = \gamma \tilde{\beta}(f) \gamma^{-1}$. So we can define a GL_n action on $\mathrm{LRep}_n(A) \times_R \mathbb{A}^n_R$ by μ: $\mathrm{GL}_n \times_R (\mathrm{LRep}_n(A) \times_R \mathbb{A}^n_R) \to \mathrm{LRep}_n(A) \times_R \mathbb{A}^n_R$ where, for any commutative R-algebra S we have

$$\begin{aligned} \mu(S)\colon \mathrm{GL}_n(S) \times (\mathrm{LRep}_n(A)(S) \times \mathbb{A}^n_R(S)) &\longrightarrow \mathrm{LRep}_n(A)(S) \times \mathbb{A}^n_R(S) \\ (\gamma, (\eta, z)) &\longmapsto (\eta^\gamma, \gamma z). \end{aligned}$$

LEMMA 1.2: *Let T be a commutative k-algebra and let (ϕ, x, P) and (ψ, z, N) represent T-points of $\mathrm{Bsev}_n(A,R)$. Let T' be a faithfully flat commutative T-algebra such that $P \otimes_T T' \cong (T')^n \cong N \otimes_T T'$ and let β_1: $P \otimes_T T' \to (T')^n$,*

β_2: $N \otimes_T T' \to (T')^n$ be T'-module isomorphisms. Then $(\phi, x, P) \sim (\psi, z, N)$ if and only if there exists a $\gamma \in GL_n(T')$ such that $\gamma(\eta_{\beta_1,\phi}, \beta_1(x)) = (\eta_{\beta_2,\psi}, \beta_2(z))$.

Proof: Assume $(\phi, x, P) \sim (\psi, z, N)$. Then there exists a T-module isomorphism $f\colon P \to N$ such that $f(x) = z$ and $\psi(a) = f\phi(a)f^{-1}$ for all $a \in A$. Let $f' = f \otimes \mathrm{id}_{T'}$. Then $(\eta_{\beta_1,\phi}, \beta_1(x))$ and $(\eta_{\beta_2 \circ f',\phi}, \beta_2 \circ f'(x)) = (\eta_{\beta_2,\psi}, \beta_2(z))$ represent the same triple (ϕ, x, P). Therefore, $\gamma = \beta_2 \circ (f')^{-1} \circ \beta_1^{-1}$ is the required element of $GL_n(T')$.

Conversely, assume that there exists a $\gamma \in \mathrm{GL}_n(T')$ such that $\gamma(\eta_{\beta_1,\phi}, \beta_1(x)) = (\eta_{\beta_2,\psi}, \beta_2(z))$. Then we claim that the restriction θ of $w \mapsto \beta_2^{-1}\gamma\beta_1(w)$ to P is an isomorphism giving the claimed equivalence. Indeed, since β_1, β_2 are isomorphisms and $\gamma \in \mathrm{GL}_n(T')$, θ must be injective. Furthermore, if $w \in P$ then since (ϕ, x, P) represents a T-point of $\mathrm{Bsev}_n(A, R)$ we know by Lemma 1.1 that there exist $a_1, \dots, a_s \in A$ and $c_1, \dots, c_s \in T$ such that $w = \sum_{i=1}^s c_i\phi(a_i)x$. Therefore

$$\begin{aligned}\theta(w) &= \sum_{i=1}^{s} c_i\theta(\phi(a_i)x)\\ &= \sum_i c_i\beta_2^{-1}(\gamma\beta_1\phi(a_i)\beta_1^{-1}\gamma^{-1})\beta_2(\beta_2^{-1}\gamma\beta_1(x))\\ &= \sum_i c_i\beta_2^{-1}(\beta_2\psi(a_i)\beta_2^{-1})\beta_2(\beta_2^{-1}(\beta_2(z)))\\ &= \sum_i c_i\psi(a_i)z,\end{aligned}$$

hence $\theta(w) \in N$.

Finally, to show θ is surjective, if $t \in N$, then by Lemma 1.1 there exist $a_1, \dots, a_s \in A$ and $c_1, \dots, c_s \in T$ such that $t = \sum_{i=1}^s c_i\psi(a_i)(z)$. Set $w = \sum_{i=1}^s c_i\phi(a_i)(x)$. Then $w \in P$ and it follows from our above calculations that $\theta(w) = t$. Therefore the map θ is an isomorphism of P and N that gives an equivalence of the triples (ϕ, x, P) and (ψ, z, N). ∎

As in [8], it is useful in our study of $\mathrm{Bsev}_n(A)$ to define the following semi-invariants of this GL_n action.

Definition 1.3: Let T be a commutative k-algebra and let $x_1, \dots, x_n \in T^n$ and let $B \in M_n(T)$ be the matrix defined by $Be_i = x_i$ for all $1 \le i \le n$ where $\{e_1, \dots, e_n\}$ is the standard basis for T^n. Then we let

$$[x_1, \dots, x_n] = \det(B).$$

Given any $a_1, \dots, a_n \in A$, we let $[a_1, \dots, a_n]$ be the morphism given by

$$\begin{aligned}[a_1, \dots, a_n]\colon \mathrm{LRep}_n(A) \times_R \mathbb{A}_R^n &\longrightarrow \mathbb{A}_R^1\\ (\eta, x) &\longmapsto [M_n(\eta)\rho_{A,n}(a_1)x, \dots, M_n(\eta)\rho_{A,n}(a_n)x].\end{aligned}$$

Note that given any $a_1, \ldots, a_n \in A$, the function

$$[a_1, \ldots, a_n] \in \Sigma_{A,n} = S_{A,n} \otimes_R R[y_1, \ldots, y_n]$$

is homogeneous of degree n. In particular, if we let $c_{i,t;j}$ denote the i,t entry of $\rho_{A,n}(a_j)$, then

$$(1) \qquad [a_1, \ldots, a_n] = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \left(\sum_{t=1}^{n} c_{1,t;\sigma(1)} y_t \right) \cdots \left(\sum_{t=1}^{n} c_{n,t;\sigma(n)} y_t \right)$$

where we use S_n to denote the symmetric group on n letters.

If ϕ: $A \to M_n(T)$ is a k-algebra homomorphism where T is a commutative k-algebra, then for any $x \in T^n$ and for any $a_1, \ldots, a_n \in A$, we will write $[a_1, \ldots, a_n](\phi, x)$ for $[a_1, \ldots, a_n](\eta_\phi, x)$.

LEMMA 1.4: *Let τ: $RF_m \to A$ be an R-algebra surjection and let η_A: $S_{m,n} \to S_{A,n}$ be the unique surjection such that $\rho_{A,n} = M_n(\eta_A) \circ \rho_{m,n}$. Then*

$$(\eta_A \otimes \mathrm{id})([H_1, \ldots, H_n]) = [\tau(H_1), \ldots, \tau(H_n)]$$

for any $H_1, \ldots, H_n \in RF_m$.

Proof: Let $\omega_{A,n} = M_n(f_{A,n}) \circ \rho_{A,n}$ where $f_{A,n}$: $S_{A,n} \to \Sigma_{A,n}$ is the canonical injection. If $\xi = (y_1, \ldots, y_n) \in (\Sigma_{A,n})^n$, then for any $a_1, \ldots, a_n \in A$ we have $[a_1, \ldots, a_n](\omega_{A,n}, \xi) = [a_1, \ldots, a_n]$ by definition. In particular, for any $H_1, \ldots, H_n \in RF_m$, $[H_1, \ldots, H_n](\omega_{m,n}, \xi) = [H_1, \ldots, H_n]$ where $\omega_{m,n} = M_n(f_{RF_m,n}) \circ \rho_{m,n}$. Since $M_n(\eta_A \otimes \mathrm{id}) \circ \omega_{m,n} = \omega_{A,n} \circ \tau$ we get

$$\begin{aligned} (\eta_A \otimes \mathrm{id})([H_1, \ldots, H_n]) &= (\eta_A \otimes \mathrm{id})([H_1, \ldots, H_n](\omega_{m,n}, \xi)) \\ &= [H_1, \ldots, H_n](M_n(\eta_A \otimes \mathrm{id}) \circ \omega_{m,n}, \xi) \\ &= [H_1, \ldots, H_n](\omega_{A,n} \circ \tau, \xi) \\ &= [\tau(H_1), \ldots, \tau(H_n)](\omega_{A,n}, \xi) \\ &= [\tau(H_1), \ldots, \tau(H_n)] \end{aligned}$$

for all $H_1, \ldots, H_n \in RF_m$. ∎

We take a little time here to note the similarity between the semi-invariants of Definition 1.3 and the functions defined in [8, Definition 1.1]. In particular, we note that the R-algebra of m generic $n \times n$ matrices can be defined as $\rho_{m,n}(RF_m)$, hence for any $H_1, \ldots, H_n \in RF_m$ it follows from the definitions that

$$[H_1, \ldots, H_n] = [\rho_{m,n}(H_1), \ldots, \rho_{m,n}(H_n)].$$

So when R is a field, the functions we define here form a subset of those defined in Definition 1.1 of [8].

More generally, for any $a_1, \ldots, a_n \in A$, Lemma 1.4 gives us

$$[a_1, \ldots, a_n] = (\eta_A \otimes \mathrm{id})([H_1, \ldots, H_n])$$

for some $H_1, \ldots, H_n \in RF_m$. Therefore many of the properties of the functions in [8, p. 857] also have analogues here. For example, given a commutative k-algebra T, for any T-point (η, x) of $\mathrm{LRep}_n(A) \times_R \mathbb{A}^n_R$ and any $\gamma \in \mathrm{GL}_n(T)$, we get

$$\begin{aligned}[a_1, \ldots, a_n](\eta^\gamma, \gamma x) &= [\gamma\phi_\eta(a_1)x, \ldots, \gamma\phi_\eta(a_n)x] \\ &= (\det(\gamma))[a_1, \ldots, a_n](\eta, x).\end{aligned} \tag{2}$$

Therefore the function $[a_1, \ldots, a_n]$ is a semi-invariant of the GL_n-action on $\mathrm{LRep}_n(A) \times_R \mathbb{A}^n_R$.

We also have the following version of Cramer's Rule.

LEMMA 1.5: *Let T be a commutative k-algebra and let $\{v_1, \ldots, v_n\}$ be a T-basis of T^n. Then for any $z \in T^n$ we have $z = \sum_{i=1}^n \alpha_i v_i$ where*

$$\alpha_i = \frac{[v_1, \ldots, v_{i-1}, z, v_{i+1}, \ldots, v_n]}{[v_1, \ldots, v_n]}$$

for all $1 \le i \le n$.

Finally, we get the analogy to [8, Theorem 1.3].

THEOREM 1.6: *Let T be a commutative k-algebra and let (ϕ, x, P) and (ψ, z, N) represent T-points of $\mathrm{Bsev}_n(A, R)$. Let T' be a faithfully flat commutative T-algebra such that $P \otimes_T T' \cong (T')^n \cong N \otimes_T T'$. Choose any T'-module isomorphisms $\beta_1\colon P \otimes_T T' \to (T')^n$, $\beta_2\colon N \otimes_T T' \to (T')^n$. Then $(\phi, x, P) \sim (\psi, z, N)$ if and only if there exists a unit $u \in T'$ such that*

$$[a_1, \ldots, a_n](\eta_{\beta_2,\psi}, \beta_2(z)) = u[a_1, \ldots, a_n](\eta_{\beta_1,\phi}, \beta_1(x))$$

for all $a_1, \ldots, a_n \in A$.

Proof: Assume $(\phi, x, P) \sim (\psi, z, N)$. Then by Lemma 1.2 there is a $\gamma \in \mathrm{GL}_n(T')$ such that $\gamma(\eta_{\beta_1,\phi}, \beta_1(x)) = (\eta_{\beta_2,\psi}, \beta_2(z))$. Then we set $u = \det(\gamma)$ and use equation (2).

Conversely, assume there is a unit $u \in T'$ such that

$$[a_1, \ldots, a_n](\eta_{\beta_2,\psi}, \beta_2(z)) = u[a_1, \ldots, a_n](\eta_{\beta_1,\phi}, \beta_1(x))$$

for all $a_1, \ldots, a_n \in A$. Since (ϕ, x, P) represents a T-point of $\mathrm{Bsev}_n(A, R)$, for each $1 \leq j \leq n$ there exist $H_{1,j}, \ldots, H_{s_j,j} \in A$ and $c_{1,j}, \ldots, c_{s_j,j} \in T'$ such that

$$\beta_1\Big(\sum_{i_j=1}^{s_j} c_{i_j,j}\phi(H_{i_j,j})x\Big) = e_j \tag{3}$$

by Lemma 1.1. For each $1 \leq j \leq n$ let

$$v_j = \beta_2\Big(\sum_{i_j=1}^{s_j} c_{i_j,j}\psi(H_{i_j,j})z\Big). \tag{4}$$

Then

$$\begin{aligned}
[v_1, \ldots, v_n] &= \sum_{i_1}\cdots\sum_{i_n}(c_{i_1,1}\cdots c_{i_n,n})[H_{i_1,1}, \ldots, H_{i_n,n}](\eta_{\beta_2,\psi}, \beta_2(z)) \\
&= u\sum_{i_1}\cdots\sum_{i_n}(c_{i_1,1}\cdots c_{i_n,n})[H_{i_1,1}, \ldots, H_{i_n,n}](\eta_{\beta_1,\phi}, \beta_1(x)) \\
&= u[e_1, \ldots, e_n] \\
&= u.
\end{aligned}$$

As u is a unit in T', the set $\{v_1, \ldots, v_n\}$ must be a T'-basis of $(T')^n$. Therefore, if we let $\gamma \in M_n(T')$ be defined by $\gamma(e_i) = v_i$ for all i, then $\gamma \in \mathrm{GL}_n(T')$ and $\det(\gamma) = u$.

Now we refer the reader to the proof of [8, Theorem 1.3] to show that $\gamma(\eta_{\beta_1,\phi}, \beta_1(x)) = (\eta_{\beta_2,\psi}, \beta_2(z))$, hence we can use Lemma 1.2 to get that $(\phi, x, P) \sim (\psi, z, N)$ as required. ∎

2. A morphism of schemes

Let $Q_{A,n} \subseteq \Sigma_{A,n}$ be the sub-R-algebra generated by the set $\{[a_1, \ldots, a_n] | a_1, \ldots, a_n \in A\}$. We will say that $h \in Q_{A,n}$ is **homogeneous of degree** q in $Q_{A,n}$ if h is a homogeneous element of $\Sigma_{A,n}$ of degree nq.

In this section we define a morphism ν: $\mathrm{Bsev}_n(A, R) \to \mathrm{Proj}(Q_{A,n})$ of R-schemes which will help clarify the correspondence between the points of $\mathrm{Bsev}_n(A, R)$ and the PGL_n-orbits of $\mathrm{LRep}_n(A) \times_R \mathbb{P}_R^{n-1}$. Later we will show that ν is actually an isomorphism (see Theorem 3.6).

First note that if $h \in Q_{A,n}$ is homogeneous of degree q, then for any commutative k-algebra T, given any T-point $(\eta, x) \in \mathrm{LRep}_n(A) \times_R \mathbb{A}_R^n$ and any $\gamma \in \mathrm{GL}_n(T)$ we have $h(\eta^\gamma, \gamma x) = (\det\gamma)^q h(\eta, x)$. Therefore, if $h' \in Q_{A,n}$ is also homogeneous of degree q, then the rational function (h/h') is constant on every GL_n-orbit for which it is defined.

THEOREM 2.1: *Let T be a commutative k-algebra and let (ϕ, x, P) represent a T-point of* $\mathrm{Bsev}_n(A, R)$. *Let T' be a commutative faithfully flat T-algebra such that $P \otimes_T T'$ is a free T'-module. Then for any isomorphism β: $P \otimes_T T' \to (T')^n$ and for any homogeneous elements $h, h' \in Q_{A,n}$ of degree q, if $h'(\eta_{\beta,\phi}, \beta(x)) \in T'$ is a unit it follows that*

$$f(\eta_{\beta,\phi}, \beta(x)) = \frac{h(\eta_{\beta,\phi}, \beta(x))}{h'(\eta_{\beta,\phi}, \beta(x))}$$

is an element of T and is independent of the choice of T' and β. In this case we just write $f(\phi, x, P)$ for $f(\eta_{\beta,\phi}, \beta(x))$.

Proof: The independence of the value of f from the choice of β follows from our discussion immediately preceding this theorem and from Theorem 1.6. So let T'' be another commutative faithfully flat T-algebra such that $P \otimes_T T''$ is a free T''-module and let β'': $P \otimes_T T'' \to (T'')^n$ be an isomorphism. Then $U = T' \otimes_T T''$ is faithfully flat over both T' and T'', hence U is faithfully flat over T. Identify T' with $T' \otimes 1 \subseteq U$ and similarly identify T'' with $1 \otimes T'' \subseteq U$. By Theorem 1.6 there exists a $\gamma \in \mathrm{GL}_n(U)$ such that $\gamma(\eta_{\beta,\phi}, \beta(x)) = (\eta_{\beta'',\phi}, \beta''(x))$. Hence $f(\eta_{\beta,\phi}, \beta(x)) = f(\eta_{\beta'',\phi}, \beta''(x))$, so the value of f is constant for each point of $\mathrm{LRep}_n(A) \times_R \mathbb{A}^n_R$ corresponding to (ϕ, x, P) and is independent of the choice of T'. Denote this value of f by $f(\phi, x, P)$.

Now consider the special case when $T'' = T'$ so $U = T' \otimes_T T'$. Then $f(\phi, x, P) \in (T' \otimes 1) \cap (1 \otimes T') \subseteq U$. So there exist $u, v \in T'$ such that $f(\phi, x, P) = u \otimes 1 = 1 \otimes v$. Let μ: $T' \otimes_T T' \to T'$ be the usual multiplication map. Then $u = \mu(f(\phi, x, P)) = v$ and so $u \otimes 1 = 1 \otimes u$. Let M be the T-submodule of T' generated by 1 and u. When we tensor the inclusion $T \subseteq M$ with T' we get that $T \otimes_T T' = M \otimes_T T'$. Since T' is faithfully flat over T, we get $T = M$ and thus $u \in T$ so $f(\phi, x, P) \in T$ as claimed. ∎

We note that the above argument that $f(\phi, x, P) \in T$ is a slight adaptation of a "faithfully flat descent" argument found in [7].

COROLLARY 2.2: *Let T be a commutative k-algebra and let (ϕ, x, P) represent a T-point p of* $\mathrm{Bsev}_n(A, R)$. *Let $f = h'/h$ for some homogeneous $h, h' \in Q_{A,n}$ such that* $\deg(h) = \deg(h')$ *and $f(\phi, x, P)$ is defined. If (ψ, z, N) of is any other representative of p then $f(\phi, x, P) = f(\psi, z, N)$.*

Proof: If $(\phi, x, P) \sim (\psi, z, N)$ then there exists an T-module isomorphism w: $P \to N$ such that $z = w(x)$ and $w\phi(a)w^{-1} = \psi(a)$ for all $a \in A$. Therefore, given a commutative faithfully flat T-algebra T' such that $N \otimes_T T'$ is a free

T'-module and an isomorphism β: $N \otimes_T T'$, we note that

$$\beta' = \beta \circ (w \otimes \mathrm{id}_{T'}) \colon P \otimes_T T' \to (T')^n$$

is also a T'-module isomorphism. Furthermore, it follows that $(\eta_{\phi,\beta'}, \beta'(x)) = (\eta_{\psi,\beta}, \beta(z))$. Therefore, by Theorem 2.1, $f(\phi, x, P) = f(\psi, z, N)$. ∎

So the degree zero homogeneous rational functions defined by elements of $Q_{A,n}$ define rational functions on $\mathrm{Bsev}_n(A, R)$. Therefore if f is a function of the type given in Corollary 2.2 and p is a T-point of $\mathrm{Bsev}_n(A, R)$ we will write $f(p)$ for the value of f at any triple representing p.

LEMMA 2.3: *Let $h_1, \ldots, h_s \in Q_{A,n}$ be homogeneous of degree 1 such that $\sum_j Q_{A,n} h_j = (Q_{A,n})_+$ where*

$$(Q_{A,n})_+ = \langle \{h \in Q_{A,n} | h \textit{ is homogeneous of positive degree } \} \rangle.$$

Then for any commutative k-algebra T and for any triple (ϕ, x, T^n) representing a T-point of $\mathrm{Bsev}_n(A, R)$, we get $\sum_j T\delta(h_j) = T$ where δ: $\Sigma_{A,n} \to T$ is the evaluation homomorphism $\delta(h) = h(\eta_\phi, x)$ for all $h \in \Sigma_{A,n}$.

Proof: For each $1 \leq j \leq s$ let Z_j be the (possibly empty) open affine subscheme of $\mathrm{Spec}(T)$ defined by $\mathrm{Spec}(T_{\delta(h_j)})$. Then the conclusion of the lemma is equivalent to saying that the Z_j form an open cover of $\mathrm{Spec}(T)$. By [4, **I**.1.7], it is sufficient to show that $\mathrm{Spec}(T)(L) = \bigcup_j Z_j(L)$ for every field L.

Let L be a field and let $v \in \mathrm{Hom}_{R\text{-alg}}(T, L) = \mathrm{Spec}_R(T)(L)$. Then $L \otimes_v T^n$ is an n-dimensional L-vector space. Furthermore, since (ϕ, x, T^n) represents a T-point of $\mathrm{Bsev}_n(A, R)$, then $M_n(v)(\phi(A))L(1 \otimes x) = L \otimes_v T^n$. So there exist $b_1, \ldots, b_n \in A$ such that $\{M_n(v)(\phi(b_1))(1 \otimes x), \ldots, M_n(v)(\phi(b_n))(1 \otimes x)\}$ forms an L-basis of $L \otimes_v T^n$. Therefore $[b_1, \ldots, b_n](M_n(v) \circ \phi, 1 \otimes x, L \otimes_v T^n) \neq 0$. Since $[b_1, \ldots, b_n] \in (Q_{A,n})_+ = \sum_j (Q_{A,n}) h_j$, there must exist a j such that $v(\delta(h_j)) = h_j(M_n(v) \circ \phi, 1 \otimes x, L \otimes_v T) \neq 0$ and hence $v \in Z_j(L)$. ∎

So we can use Lemma 2.3 to define a morphism ν: $\mathrm{Bsev}_n(A, R) \to \mathrm{Proj}(Q_{A,n})$ as follows. Let T be any commutative k-algebra and let p be a T-point of $\mathrm{Bsev}_n(A, R)$ represented by the triple (ϕ, x, P). Choose a faithfully flat finitely presented commutative T-algebra T' such that $P \otimes_T T'$ is free and choose a T'-module isomorphism β: $P \otimes_T T' \to (T')^n$. Identify T with an appropriate sub-R-algebra of T' and let δ_β: $\Sigma_{A,n} \to T'$ be the evaluation homomorphism given by $\delta_\beta(h) = h(\eta_{\beta,\phi}, \beta(x))$ for all $h \in \Sigma_{m,n}$.

Let $\{h_1, \ldots, h_s\} \subseteq Q_{A,n}$ be a set of homogeneous elements of degree 1 such that $\sum_j (Q_{A,n}) h_j = (Q_{A,n})_+$. Now, by Lemma 2.3, the subschemes $Z'_j =$

$\mathrm{Spec}(T'_{\delta_\beta(h_j)}) \subseteq \mathrm{Spec}(T')$ form an open affine cover of $\mathrm{Spec}(T')$. For each j we can define a morphism $\theta_j\colon Z'_j \to \mathrm{Spec}((Q_{A,n})_{((h_j))})$ where the comorphism of θ_j is the R-algebra homomorphism $w_j\colon (Q_{A,n})_{((h_j))} \to T'_{\delta_\beta(h_j)}$ induced from δ_β. Since the Z'_j form an affine open cover of $\mathrm{Spec}(T')$ and the w_j are induced from the R-algebra homomorphism δ_β, the θ_j will glue together to define a morphism $\theta\colon \mathrm{Spec}(T') \to \mathrm{Proj}(Q_{A,n})$.

By our choice of T', the canonical morphism $f\colon \mathrm{Spec}(T') \to \mathrm{Spec}(T)$ is a faithfully flat morphism of finite type. Therefore, by [5, Proposition I.2.7 and Theorem I.2.12] for example, f is surjective and open. Therefore the images $f(Z_j)$ form an open cover of $\mathrm{Spec}(T)$. For each $1 \leq j \leq s$ choose elements $a_{i,j} \in T$ such that the $U_{i,j} = \mathrm{Spec}(T_{a_{i,j}})$ form an open cover of $f(Z'_j)$. Then $T'_{a_{i,j}}$ is faithfully flat over $T_{a_{i,j}}$ and the $U'_{i,j} = \mathrm{Spec}(T'_{a_{i,j}})$ form an open affine cover of $\mathrm{Spec}(T')$.

So for every i and j let $w_{i,j}\colon (Q_{A,n})_{((h_j))} \to T'_{a_{i,j}}$ be the composition of ω_j with the canonical homomorphism $T'_{\delta_\beta(h_j)} \to T'_{a_{i,j}}$. Then by Theorem 2.1 $w_{i,j}((Q_{A,n})_{((h_j))}) \subseteq T_{a_{i,j}}$. Therefore θ actually defines a T-point of $\mathrm{Proj}(Q_{A,n})$ which we will call $\nu(T)(p)$.

THEOREM 2.4: *The transformation $\nu\colon \mathrm{Bsev}_n(A,R) \to \mathrm{Proj}(Q_{A,n})$ given above defines a natural transformation of functors, hence a morphism of R-schemes.*

Proof: Given our above discussion, the only thing left to prove is that for any homomorphism of commutative k-algebras $f\colon T \to U$ we have

$$\mathrm{Proj}(Q_{A,n})(f) \circ \nu(T) = \nu(U) \circ \mathrm{Bsev}_n(A,R)(f).$$

But we can use faithfully flat descent to reduce the proof to showing this equality holds for all the T-points of $\mathrm{Bsev}_n(A,R)$ that can be represented by triples of the form (ϕ, x, T^n).

Let (ϕ, x, T^n) represent a T-point p of $\mathrm{Bsev}_n(A,R)$. Then $\mathrm{Bsev}_n(A,R)(f)(p)$ is represented by the triple

$$(M_n(f)\phi, x \otimes_f 1, T^n \otimes_f U \cong U^n)$$

so $\nu(U)\circ\mathrm{Bsev}_n(A,R)(f)(p)$ is the U-point of $\mathrm{Proj}(Q_{A,n})$ induced from $\delta_1\colon \Sigma_{A,n} \to U$ given by $\delta_1(h) = h(\eta_{M_n(f)\phi}, x \otimes 1)$. Similarly, $\mathrm{Proj}(Q_{A,n})(f) \circ \nu(T)(p)$ is the point induced by $\delta_2\colon \Sigma_{A,n} \to U$ where $\delta_2 = f \circ \delta$ and $\delta\colon \Sigma_{A,n} \to T$ is given by $\delta(h) = h(\eta_\phi, x)$. Clearly $\delta_1 = \delta_2$ so we get the desired equality and hence ν is a morphism of schemes. ∎

3. An isomorphism of schemes

In this section we will prove that the morphism ν: $\mathrm{Bsev}_n(A,R) \to \mathrm{Proj}(Q_{A,n})$ is an isomorphism of k-schemes. We note that to prove that ν is an isomorphism, it is sufficient to show that for every commutative k-algebra T that $\nu(T)$ is a bijection (e.g., [3, Exercise X.1.6]). As Theorem 1.6 already tells us that such a $\nu(T)$ is injective, in this section we will show that $\nu(T)$ is also surjective.

Let T be an arbitrary k-algebra. To show that $\nu(T)$ is surjective, we will first construct locally free sheaf on $\mathrm{Proj}(Q_{A,n})$ which is in some sense a universal sheaf of our PGL_n-quotient scheme. Our starting point is an analogy to the argument given on pages 861–862 of [8]. For the reader's convenience, we review Van den Bergh's definition of a special sequence.

An (m,n)**-special sequence** is a sequence of $(n-1)$ ordered integer pairs $\{(\alpha_j,\beta_j)\}_{j=2}^n$ such that $1 \le \beta_j \le m$, $1 \le \alpha_j < j$ for all $2 \le j \le n$ and $j \ne j'$ implies $(\alpha_j,\beta_j) \ne (\alpha_{j'},\beta_{j'})$. When m and n are understood, we will just call M a special sequence.

For each special sequence $M = \{(\alpha_j,\beta_j)\}_{j=2}^n$, we inductively define a sequence of monomials $H_1^{(M)},\ldots,H_n^{(M)} \in RF_m$ by letting $H_1^{(M)} = 1$ and $H_j^{(M)} = \mathcal{Y}_{\beta_j}H_{\alpha_j}^{(M)}$ for $2 \le j \le n$. Let $h_M = [H_1^{(M)},\ldots,H_n^{(M)}]$. Now, for each $1 \le j \le n$ let $a_j^{(M)} = \tau(H_j^{(M)})$ and define $g_M = [a_1^{(M)},\ldots,a_n^{(M)}] \in Q_{A,n}$. Let $V_M = \mathrm{Spec}((Q_{A,n})_{((g_M))})$ for each special sequence M.

LEMMA 3.1: *The set of subschemes $\{V_M | M$ is special$\}$ of $\mathrm{Proj}(Q_{A,n})$ forms an open affine covering of $\mathrm{Proj}(Q_{A,n})$.*

Proof: This follows directly from [4, I.1.7] and [8, Lemma 1.6]. ∎

Let τ: $RF_m \to A$ be an R-algebra surjection. For each special sequence M we can define the analogy to the coordinate functions given in Equation (3) on p. 861 of [8]. So for each $1 \le i,j \le n$, $1 \le \ell \le m$ let

$$t_{i,j,\ell}^{(M)} = [H_1^{(M)},\ldots,H_{i-1}^{(M)},\mathcal{Y}_\ell H_j^{(M)},H_{i+1}^{(M)},\ldots,H_n^{(M)}]$$

and let

$$u_{i,j,\ell}^{(M)} = (\eta_A \otimes \mathrm{id})(t_{i,j;\ell}^{(M)}) = [\ldots,a_{i-1}^{(M)},\tau(\mathcal{Y}_\ell)a_j^{(M)},a_{i+1}^{(M)},\ldots]$$

where we use η_A: $S_{m,n} \to S_{A,n}$ to denote the unique surjection such that $\rho_{A,n} \circ \tau = M_n(\eta_A) \circ \rho_{m,n}$.

LEMMA 3.2: *Let M be a special sequence. Then $R[V_M] = (Q_{A,n})_{((g_M))}$ is generated as an R-algebra by the set*

$$W_M = \{w_{i,j;\ell}^{(M)} g_M^{-1} | 1 \leq i,j \leq n, 1 \leq \ell \leq m\}.$$

Proof: Let Q_1 be the sub-R-module of $Q_{A,n}$ consisting of the zero element and all homogeneous elements of degree 1 in $Q_{A,n}$. Let $Q_1 g_M^{-1} = \{h g_M^{-1} | h \in Q_1\}$. As $Q_{A,n}$ is generated by Q_1 as an R-algebra, it suffices to show that the R-module generated by W_M is equal to $Q_1 g_M^{-1}$.

Clearly $\sum_{i,j,\ell} R w_{i,j;\ell}^{(M)} \subseteq Q_1 g_M^{-1}$. Therefore we can use [6, Thm. 3.80], for example, to reduce to the case when R is a local ring with maximal ideal $\boldsymbol{m}$. We remark that $Q_1 g_M^{-1}$ is a finitely generated R-module. Let $K = R/\boldsymbol{m}$ and let $\bar{K}$ be its algebraic closure. Then it follows as an easy corollary of [8, Thm. 1.9] that

$$\Big(\sum_{i,j,\ell} R w_{i,j;\ell}^{(M)} g_M^{-1}\Big) \otimes_R \bar{K} = Q_1 g_M^{-1} \otimes_R \bar{K}.$$

As $\bar{K}$ is faithfully flat over K, we get

$$\Big(\sum_{i,j,\ell} R w_{i,j;\ell}^{(M)} g_M^{-1}\Big) \otimes_R K = Q_1 g_M^{-1} \otimes_R K.$$

Hence, by Nakayama's Lemma,

$$\Big(\sum_{i,j,\ell} R w_{i,j;\ell}^{(M)} g_M^{-1}\Big) = Q_1 g_M^{-1}$$

so we are done. ∎

For each special sequence, define $\psi_M \colon S_{m,n} \to (Q_{m,n})_{((h_M))}$ by letting $\psi_M(x_{i,j}^{(\ell)}) = t_{i,j;\ell}^{(M)} h_M^{-1}$. Let $\Psi_M = M_n(\psi_M) \circ \rho_{m,n}$. For any commutative k-algebra T and any k-algebra homomorphism $\eta \colon S_{m,n} \to T$, let

$$\eta \otimes \mathrm{id} = \eta \otimes \mathrm{id} \colon \Sigma_{m,n} \to T[y_1, \ldots, y_n]$$

be the induced graded homomorphism. Then for any homogeneous $h \in \Sigma_{m,n}$, $\eta \otimes \mathrm{id}$ induces a homomorphism $\eta_{(h)} \colon (\Sigma_{m,n})_{((h))} \to (T[y_1, \ldots, y_n])_{((\eta \otimes \mathrm{id}(h)))}$.

LEMMA 3.3: *Let $\tau \colon RF_m \to A$ be an R-algebra surjection and let $\eta \colon S_{m,n} \to S_{A,n}$ be the unique surjection such that $\rho_{A,n} \circ \tau = M_n(\eta) \circ \rho_{m,n}$. Then for any $H_1, \ldots, H_n \in RF_m$ and for any special sequence M,*

$$[H_1, \ldots, H_n](M_n(\eta_{(h_M)}) \circ \Psi_M, e_1) = \frac{[\tau(H_1), \ldots, \tau(H_n)]}{g_M}.$$

Proof: Let $M = \{(\alpha_j, \beta_j) | 2 \leq j \leq n\}$ be a special sequence. Then for each $1 \leq i, j \leq n$

$$t^{(M)}_{i,\alpha_j;\beta_j} = [\ldots, H^{(M)}_{i-1}, \mathcal{Y}_{\beta_j} H^{(M)}_{\alpha_j}, H^{(M)}_{i+1}, \ldots] = \begin{cases} h_M & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

since $H^{(M)}_j = \mathcal{Y}_{\beta_j} H^{(M)}_{\alpha_j}$. As $H^{(M)}_1 = 1$, we have $\Psi_M(H^{(M)}_1)e_1 = e_1$. Therefore for $j \geq 2$ we get can use induction to get $\Psi_M(H^{(M)}_j)(e_1) = \Psi_M(\mathcal{Y}_{\beta_j})\Psi_M(H^{(M)}_{\alpha_j})e_1 = \Psi_M(\mathcal{Y}_{\beta_j})e_{\alpha_j} = e_j$. Hence

$$\begin{aligned} t^{(M)}_{i,j;\ell}&(M_n(\eta_{(h_M)}) \circ \Psi_M, e_1) \\ &= \eta_{(h_M)}[\ldots, \Psi_M(H^{(M)}_{i-1})e_1, \Psi_M(\mathcal{Y}_\ell)\Psi_M(H^{(M)}_j)e_1, \Psi_M(H^{(M)}_{i+1})e_1, \ldots] \\ &= \eta_{(h_M)}[\ldots, e_{i-1}, \Psi_M(\mathcal{Y}_\ell)e_j, e_{i+1}, \ldots] \\ &= \eta_{(h_M)}(i, j\text{th entry of } \Psi_M(\mathcal{Y}_\ell)) \\ &= \eta_{(h_M)}(\psi_M(x^{(\ell)}_{i,j})) \\ &= \eta_{(h_M)}(t^{(M)}_{i,j;\ell} h_M^{-1}) \\ &= w^{(M)}_{i,j;\ell} g_M^{-1} \\ &= \frac{[\tau(H^{(M)}_1), \ldots, \tau(H^{(M)}_n)]}{g_M}. \end{aligned}$$

In particular, $h_M(M_n(\eta_{(h_M)}) \circ \Psi_M, e_1) = 1$ and so for any $H_1, \ldots, H_n \in RF_m$

$$[H_1, \ldots, H_n](M_n(\eta_{(h_M)}) \circ \Psi_M, e_1) = \frac{[H_1, \ldots, H_n]}{h_M}(M_n(\eta_{(h_M)}) \circ \Psi_M, e_1)$$

But Lemma 3.2 tells us that $[H_1, \ldots, H_n]h_M^{-1} \in (Q_{m,n})_{((h_M))}$ can be expressed in terms of the $t^{(M)}_{i,j;\ell} h_M^{-1}$. Hence for any $H_1, \ldots, H_n \in RF_M$, the value of

$$\frac{[H_1, \ldots, H_n]}{h_M}(M_n(\eta_{(h_M)}) \circ \Psi_M, e_1)$$

is completely determined by the values $t^{(M)}_{i,j;\ell}(M_n(\eta_{(h_M)}) \circ \Psi_M, e_1)$. So the result follows. ∎

LEMMA 3.4: *Let τ: $RF_m \to A$ be a surjection and let η: $S_{m,n} \to S_{A,n}$ be the unique homomorphism such that $\rho_{A,n} \circ \tau = M_n(\eta) \circ \rho_{m,n}$. Then for any special sequence M there exists a homomorphism Φ_M: $A \to M_n(R[V_M])$ such that $\Phi_M \circ \tau = M_n(\eta_{(h_M)}) \circ \Psi_M$.*

Proof: Let

$$\phi_M \colon S_{A,n} \cong (S_{A,n} \otimes_\eta S_{m,n}) \to (S_{A,n} \otimes_\eta (\Sigma_{m,n})_{((h_M))}) \cong (\Sigma_{A,n})_{((g_M))}$$

be the homomorphism induced by tensoring

$$\psi_M\colon S_{m,n} \to (Q_{m,n})_{((h_M))} \subseteq (\Sigma_{m,n})_{((h_M))}$$

with $S_{A,n}$ over η. Then $\phi_M \circ \eta = \eta_{(h_M)} \circ \psi_M$. Therefore

$$\begin{aligned} M_n(\eta_{(h_M)}) \circ \Psi_M &= M_n(\eta_{(h_M)}) \circ M_n(\psi_M) \circ \rho_{m,n} \\ &= M_n(\phi_M) \circ M_n(\eta) \circ \rho_{m,n} \\ &= M_n(\phi_M) \circ \rho_{A,n} \circ \tau. \end{aligned}$$

Let $\Phi_M = M_n(\phi_M) \circ \rho_{A,n}$. Then Φ_M has the desired property. ∎

Note that for any special sequence M the triple

$$(M_n((\eta_A)_{(h_M)}) \circ \Psi_M, e_1, (R[V_M])^n)$$

represents an $R[V_M]$-point of $\mathrm{Bsev}_n(RF_m, R)$. Indeed, since h_M evaluated at this triple is necessarily 1, the set

$$\{M_n((\eta_A)_{(h_M)})\Psi_M(H_1^{(M)})e_1, \ldots, M_n((\eta_A)_{(h_M)})\Psi_M(H_n^{(M)})e_1\}$$

forms an $R[V_M]$-basis of $(R[V_M])^n$. Therefore, by Lemma 3.4, the triple $(\Phi_M, e_1, (R[V_M])^n)$ represents an $R[V_M]$-point of $\mathrm{Bsev}_n(A, R)$.

Let M and M' be any two special sequences, and let $\lambda_{M,M'}\colon R[V_M] = (Q_{A,n})_{((g_M))} \to (Q_{A,n})_{((g_M g_{M'}))}$ be the canonical homomorphism. Then by Lemma 3.3 for any $a_1, \ldots, a_n \in A$ we get

$$[a_1, \ldots, a_n](M_n(\lambda_{M,M'})\Phi_M, e_1) = (h_{M'}/h_M)[a_1, \ldots, a_n](M_n(\lambda_{M',M})\Phi_{M'}, e_1).$$

Therefore, by Theorem 1.6, there exists a $\gamma_{M',M} \in \mathrm{GL}_n((Q_{A,n})_{((g_M g_{M'}))})$ such that

$$\gamma_{M',M}(M_n(\lambda_{M,M'})\Phi_M, e_1) = (M_n(\lambda_{M',M})\Phi_{M'}, e_1).$$

Now we can use [2, Exer. II.1.22], for example, to glue these triples into a triple $(\Phi, \sigma, \mathcal{L})$ such that $\mathcal{L}$ is a locally free sheaf of rank n on $\mathrm{Proj}(Q_{A,n})$, Φ is a global section of the sheaf of homomorphisms $\mathrm{Hom}(A, \mathrm{End}(\mathcal{L}(\_)))$, and σ is a global section of $\mathcal{L}$.

PROPOSITION 3.5: *Let T be a commutative k-algebra. Then for every point p: $\mathrm{Spec}(T) \to \mathrm{Proj}(Q_{A,n})$ the triple $(p^*\Phi, p^*\sigma, \Gamma(p^*\mathcal{L}))$ represents a T-point of $\mathrm{Bsev}_n(A, R)$. (Here we use $\Gamma(\_)$ to denote the global section of a sheaf.)*

Proof: First note since $\mathcal{L}$ is a locally free sheaf of rank n on $\mathrm{Proj}(Q_{A,n})$ that $p^*\mathcal{L}$ is locally free of rank n on $\mathrm{Spec}(T)$. Furthermore, by Lemma 3.2, for any open subscheme V of $\mathrm{Proj}(Q_{A,n})$, $\mathcal{L}(V)$ is finitely generated as an $\mathcal{O}_{Q_{A,n}}(V)$-module, where $\mathcal{O}_{Q_{A,n}}$ denotes the structure sheaf of $\mathrm{Proj}(Q_{A,n})$. So $\Gamma(p^*\mathcal{L})$ is finitely generated as a T-module. Therefore $\Gamma(p^*\mathcal{L})$ is a projective T-module of constant rank n. Also $p^*\sigma \in \Gamma(p^*\mathcal{L})$ by definition. Finally, we need to show that $p^*\Phi(A)T(p^*\sigma) = \Gamma(p^*\mathcal{L})$.

By [6, Thm. 3.80], for example, we can assume T is a local R-algebra. Therefore by Lemma 3.1 there exists a special sequence M such that $g_M(p^*\Phi, p^*\sigma)$ is a unit in T. Hence the set $\{p^*\Phi(a_1^{(M)})p^*\sigma, \ldots, p^*\Phi(a_n^{(M)})p^*\sigma\}$ defines a T-basis of $\Gamma(p^*\mathcal{L})$. Therefore the triple $(p^*\Phi, p^*\sigma, \Gamma(p^*\mathcal{L}))$ defines a T-point of $\mathrm{Bsev}_n(A, R)$. ∎

THEOREM 3.6: *For any commutative k-algebra T the map $\nu(T)$ is a surjection. Therefore the morphism ν: $\mathrm{Bsev}_n(A, R) \to \mathrm{Proj}(Q_{A,n})$ is an isomorphism.*

Proof: For any commutative k-algebra T, Theorem 1.6 implies $\nu(T)$ is injective. Therefore if we show $\nu(T)$ is also surjective we can define an inverse morphism ν^{-1} to ν by letting $\nu^{-1}(T) = \nu(T)^{-1}$ for every commutative k-algebra T (e.g., [3, Exercise X.1.6]).

Let T be an arbitrary commutative k-algebra and let p: $\mathrm{Spec}(T) \to \mathrm{Proj}(Q_{A,n})$ be a T-point of $\mathrm{Proj}(Q_{A,n})$. Then, by Proposition 3.5, the triple $(p^*\Phi, p^*\sigma, \Gamma(p^*\mathcal{L}))$ represents a T-point q of $\mathrm{Bsev}_n(A, R)$. Therefore, by showing $\nu(T)(q) = p$ we prove the theorem.

Let $\Lambda \subseteq T$ be such that the set $\{\mathrm{Spec}(T_w) | w \in \Lambda\}$ is an open cover of $\mathrm{Spec}(T)$, for every $w \in \Lambda$, $P_w = P \otimes_T T_w$ is a free T_w-module, and $\mathrm{Spec}(T_w)$ is a subscheme of $p^{-1}(V_M)$ for some special sequence M. Let $T' = \prod_{w\in\Lambda} T_w$. Then T' is a faithfully flat T-algebra and $P \otimes_T T'$ is a free T'-module. Let j: $\mathrm{Spec}(T') \to \mathrm{Spec}(T)$ be the morphism defined by the diagonal homomorphism $T \hookrightarrow T'$ given by $t \mapsto \prod_{w\in\Lambda} t_w$. So by showing $\nu(T)(q) \circ j = p \circ j$, we see that $\nu(T)(q)$ and p agree locally, hence are equal as required.

Since $P \otimes_T T'$ is a free T'-module, $\nu(T)(q) \circ j$ is defined by the evaluation homomorphism ξ: $Q_{A,n} \to T'$ determined by

$$\begin{aligned} \xi([a_1, \ldots, a_n]) &= [a_1, \ldots, a_n]((p\circ j)^*\Phi, (p\circ j)^*\sigma, \Gamma((p\circ j)^*\mathcal{L}) \\ &= \textstyle\prod_{w\in\Lambda}[a_1, \ldots, a_n]((p\circ i_w)^*\Phi, (p\circ i_w)^*\sigma, \Gamma((p\circ i_w)^*\mathcal{L}) \end{aligned}$$

for any $a_1, \ldots, a_n \in A$ where i_w: $\mathrm{Spec}(T_w) \to \mathrm{Spec}(T)$ denotes the canonical embedding.

Choose $w \in \Lambda$ and let M be a special sequence such that $\mathrm{Spec}(T_w)$ is a subscheme of $p^{-1}(V_M)$. Let p_w: $\mathrm{Spec}(T_w) \to V_M$ be the morphism induced by p, let α_w: $R[V_M] \to T_w$ be the co-morphism of p_w, and let ξ_w: $R[V_M] \to T_w$ be the homomorphism induced from ξ. If i_M: $V_M \to \mathrm{Proj}(Q_{A,n})$ is the canonical embedding, then $i_M \circ p_w = p \circ i_w$. So for any $a_1, \ldots, a_n \in A$,

$$\begin{aligned}\xi([a_1,\ldots,a_n]g_M^{-1}) &= ([a_1,\ldots,a_n]g_M^{-1})((p\circ i_w)^*\Phi,(p\circ i_w)^*\sigma,\Gamma((p\circ i_w)^*\mathcal{L}))\\ &= ([a_1,\ldots,a_n]g_M^{-1})((i_M\circ p_w)^*\Phi,(i_M\circ p_w)^*\sigma,\Gamma((i_M\circ p_w)^*\mathcal{L}))\\ &= ([a_1,\ldots,a_n]g_M^{-1})(M_n(\alpha_w)\Phi_M,e_1\otimes 1,(R[V_M])^n\otimes_{\alpha_w}T_w)\\ &= \alpha_w(([a_1,\ldots,a_n]g_M^{-1})(\Phi_M,e_1,(R[V_M])^n))\\ &= \alpha_w([a_1,\ldots,a_n]g_M^{-1})\end{aligned}$$

where the last equality follows from Lemmas 3.3 and 3.4. Since $R[V_M]$ is generated by the elements $[a_1,\ldots,a_n]g_M^{-1}$ (by Lemma 3.2), we get that α_w and ξ_w agree for an arbitrary $w \in \Lambda$. Hence we conclude that $\nu(T)(q) \circ j = p \circ j$ as required. ∎

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